CAS-HEP-T-96-03/004, OHSTPY-HEP-T-96-008

hep-ph/9603282

# A Predictive SUSY SO(10)$\times\Delta(48)\times$ U(1) Model for CP Violation, Neutrino Oscillation, Fermion Masses and Mixings with Very Low $\tan\beta$

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# A Predictive SUSY SO(10)×Δ(48)× U(1) Model for CP Violation, Neutrino Oscillation, Fermion Masses and Mixings with Very Low $\tan\beta$

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(March 1996, hep-ph/9603282)

## Abstract

Assuming universality of Yukawa coupling of the superpotential and maximal spontaneous CP violation, fermion masses and mixing angles including that of neutrinos are studied in an SUSY SO(10)×Δ(48)× U(1) model with small $\tan\beta$. The low energy parameters of the standard model are determined solely by the Clebsch factors of the symmetry group and the structure of the physical vacuum. Thirteen parameters involving masses and mixing angles in the quark and charged lepton sector are successfully predicted by only four parameters with three of them determined by the scales of U(1)×Δ(48), SO(10), SU(5) and SU(2)$_L$ symmetry breakings. An interesting prediction on ten parameters concerning the neutrino sector is also made by using the same four parameters. An additional parameter is added to obtain the mass and mixing of a sterile neutrino. It is found that the LSND $\bar{\nu}_\mu \rightarrow \bar{\nu}_e$ events,

[*]supported in part by US Department of Energy Grant# DOE/ER/01545-675

atmospheric neutrino deficit and the mass limit put by hot dark matter can be naturally explained. Solar neutrino puzzle can be solved only by introducing a sterile neutrino. $(\nu_e - \nu_\tau)$ oscillation is found to have the same sensitive region as the $(\nu_e - \nu_\mu)$ oscillation. The hadronic parameters $B_K$ and $f_B\sqrt{B}$ are extracted from the observed $K^0$-$\bar{K}^0$ and $B^0$-$\bar{B}^0$ mixings respectively. The direct CP violation $(\varepsilon'/\varepsilon)$ in kaon decays and the three angles $\alpha$, $\beta$ and $\gamma$ of the unitarity triangle in the CKM matrix are also presented. More precise measurements of $\alpha_s(M_Z)$, $|V_{cb}|$, $|V_{ub}/V_{cb}|$, $m_t$, as well as various CP violation and neutrino oscillation experiments will provide crucial tests for the present model.

# I. INTRODUCTION

The standard model (SM) is a great success. Eighteen phenomenological parameters in the SM, which are introduced to describe all the low energy data in the quark and charged lepton sector, have been extracted from various experiments although they are not yet equally well known. Some of them have an accurcy of better than 1%, but some others less than 10%. To improve the accuracy for these parameters and understand them is a big challenge for particle physics. The mass spectrum and the mixing angles observed remind us that we are in a stage similar to that of atomic spectroscopy before Balmer. Much effort has been made along this direction. It was first observed by Gatto *et al*, Cabbibo and Maiani [1] that the Cabbibo angle is close to $\sqrt{m_d/m_s}$. This observation initiated the investigation of the texture structure with zero elements [2] in the fermion Yukawa coupling matrices. The well-known examples are the Fritzsch ansatz [3] and Georgi-Jarlskog texture [4], which has been extensively studied and improved substantially in the literature [5]. Ramond, Robert and Ross [6] presented recently a general analysis on five symmetric texture structures with zeros in the quark Yukawa coupling matrices. A general analysis and review of the previous studies on the texture structure was given by Raby in [7]. Recently, Babu and Barr [8], Babu and Mohapatra [9], Babu and Shafi [10], Hall and Raby [11], Berezhiani [12], Kaplan and Schmaltz [13], Kusenko and Shrock [14] constructed some interesting models with texture zeros based on supersymmetric (SUSY) SO(10). Anderson, Dimopoulos, Hall, Raby and Starkman [15] presented a general operator analysis for the quark and charged lepton Yukawa coupling matrices with two zero textures '11' and '13'. Though the texture '22' and '32' are not unique they could fit successfully the 13 observables in the quark and charged lepton sector with only six parameters. Recently, we have shown [16] that the same 13 parameters as well as 10 parameters concerning the neutrino sector (though not unique for this sector) can be successfully described in an SUSY SO(10)$\times\Delta(48)\times$ U(1) model with large $\tan\beta$, where the universality of Yukawa coupling of superpotential was assumed. The resulting texture of mass matrices in the low energy region is quite unique and depends

only on a single coupling constant and some vacuum expectation values (VEVs) caused by necessary symmetry breaking. The 23 parameters were predicted by only five parameters with three of them determined by the symmetry breaking scales of U(1), SO(10), SU(5) and $SU(2)_L$. In that model, the ratio of the VEVs of two light Higgs $\tan\beta \equiv v_2/v_1$ has large value $\tan\beta \sim m_t/m_b$. In general, there exists another interesting solution with small value of $\tan\beta \sim 1$. Such a class of model could also give a consistent prediction on top quark mass and other low energy parameters. Furthermore, models with small value of $\tan\beta \sim 1$ are of phenomenological interest in testing Higgs sector in the minimum supersymmetric standard model (MSSM) at the Colliders [17]. Most of the existing models with small values of $\tan\beta$ in the literature have more parameters than those with large values of $\tan\beta \sim m_t/m_b$. This is because the third family unification condition $\lambda_t^G = \lambda_b^G = \lambda_\tau^G$ has been changed to $\lambda_t^G \neq \lambda_b^G = \lambda_\tau^G$. Besides, some relations between the up-type and down-type quark (or charged lepton) mass matrices have also been lost in the small $\tan\beta$ case when two light Higgs doublets needed for $SU(2)_L$ symmetry breaking belong to different 10s of SO(10). Although models with large $\tan\beta$ have less parameters, large radiative corrections [18] to the bottom quark mass and Cabibbo-Kobayashi-Maskawa (CKM) mixing angles might arise depending on an unkown spectrum of supersymmetric particles.

In a recent Rapid Communication [19], we have presented an alternative model with small value of $\tan\beta \sim 1$ based on the same symmetry group SUSY SO(10)$\times\Delta(48)\times$U(1) as the model [16] with large value of $\tan\beta$. It is amazing to find out that the model with small $\tan\beta \sim 1$ in [19] has more predictive power on fermion masses and mixings. For convenience, we refer the model in [16] as Model I (with large $\tan\beta \sim m_t/m_b$) and the model in [19] as Model II (with small $\tan\beta \sim 1$).

In this paper, we will present in much greater detail an analysis for the model II. Our main considerations can be summarized as follows:

1) The non-abelian dihedral group $\Delta(48)$, a subgroup of SU(3) ($\Delta(3n^2)$ with $n = 4$), is taken as the family group. U(1) is family-independent and is introduced to distinguish various fields which belong to the same representations of SO(10)$\times\Delta(48)$. The irreducible

representations of $\Delta(48)$ consisting of five triplets and three singlets are found to be sufficient to build interesting texture structures for fermion mass matrices. The symmetry $\Delta(48)\times$ U(1) naturally ensures the texture structure with zeros for fermion Yukawa coupling matrices. Furthermore, the non-abelian flavor symmetries provides a super-GIM mechanism to supress flavor changing neutral currents induced by supersymmetric particles [20,21,13,22].

2) The universality of Yukawa coupling of the superpotential before symmetry breaking is assumed to reduce possible free parameters, i.e., all the coupling coefficients are assumed to be equal and have the same origins from perhaps a more fundamental theory. We know in general that universality of charges occurs only in the gauge interactions due to charge conservation like the electric charge of different particles. In the absence of strong interactions, family symmetry could keep the universality of weak interactions in a good approximation after breaking. In theories of the present kind, there are very rich structures above the grand unification theory (GUT) scale with many heavy fermions and scalars and their interactions are taken to be universal at the GUT scale where family symmetries have been broken. All heavy fields must have some reasons to exist and interact which we do not understand at this moment. So that it can only be an ansatz at the present moment since we do not know the answer governing the behavior of nature above the GUT scale. As the numerical predictions on the low energy parameters so found are very encouraging and interesting, we believe that there must be a deeper reason that has to be found in the future.

3) The two light Higgs doublets are assumed to belong to an unique 10 representation Higgs of SO(10).

4) Both the symmetry breaking direction of SO(10) down to SU(5) and the two symmetry breaking directions of SU(5) down to $SU(3)_c\times SU(2)_L\times U(1)$ are carefully chosen to ensure the needed Clebsch coefficients for quark and lepton mass matrices . The mass splitting between the up-type quark and down-type quark (or charged lepton) Yukawa couplings is attributed to the Clebsch factors caused by the SO(10) symmetry breaking direction. Thus the third family four-Yukawa coupling relation at the GUT scale will be given by

$$\lambda_b^G = \lambda_\tau^G = \frac{1}{3^n}\lambda_t^G = 5^{n+1}\lambda_{\nu_\tau}^G \tag{1}$$

where the factors $1/3^n$ and $5^{n+1}$ with $n$ being an integer are the Clebsch factors. A factor $1/3^n$ will also multiply the down-type quark and charged lepton Yukawa coupling matrices.

5) CP symmetry is broken spontaneously in the model, a maximal CP violation is assumed to further diminish free parameters.

With the above considerations, the resulting model has found to provide a successful prediction on 13 parameters in the quark and charged lepton sector as well as an interesting prediction on 10 parameters in the neutrino sector with only four parameters. One is the universal coupling constant and the other three are determined by the vacuum expectation values (VEVs) of the symmetry breaking scales. One additional parameter resulted from the VEV of a singlet scalar is introduced to obtain the mass and mixing angle of a sterile neutrino. Our paper is organized as follows: In section 2, we will present the results of the Yukawa coupling matrices. The resulting masses and CKM quark mixings are presented in section 3. In section 4 neutrino masses and CKM-type mixings in the lepton sector are presented. All existing neutrino experiments are discussed and found to be understandable in the present model. In section 5, the representations of the dihedral group $\Delta(48)$ and their tensor products are explicitly presented. In section 6, the model with superfields and superpotential is constructed in detail. Conclusions and remarks are presented in the last section.

## II. YUKAWA COUPLING MATRICES

With the above considerations, a model based on the symmetry group SUSY SO(10)$\times\Delta(48)\times$ U(1) with a single coupling constant and small value of $\tan\beta$ is constructed. Yukawa coupling matrices which determine the masses and mixings of all quarks and leptons are obtained by carefully choosing the structure of the physical vacuum. We find

$$\Gamma_u^G = \frac{2}{3}\lambda_H \begin{pmatrix} 0 & \frac{3}{2}z'_u\epsilon_P^2 & 0 \\ \frac{3}{2}z_u\epsilon_P^2 & -3y_u\epsilon_G^2 e^{i\phi} & -\frac{\sqrt{3}}{2}x_u\epsilon_G^2 \\ 0 & -\frac{\sqrt{3}}{2}x_u\epsilon_G^2 & w_u \end{pmatrix} \tag{2}$$

and

$$\Gamma_f^G = \frac{2}{3}\lambda_H \frac{(-1)^{n+1}}{3^n} \begin{pmatrix} 0 & -\frac{3}{2}z'_f\epsilon_P^2 & 0 \\ -\frac{3}{2}z_f\epsilon_P^2 & 3y_f\epsilon_G^2 e^{i\phi} & -\frac{1}{2}x_f\epsilon_G^2 \\ 0 & -\frac{1}{2}x_f\epsilon_G^2 & w_f \end{pmatrix} \tag{3}$$

for $f = d, e$, and

$$\Gamma_\nu^G = \frac{2}{3}\lambda_H \frac{1}{5}\frac{(-1)^{n+1}}{15^n} \begin{pmatrix} 0 & -\frac{15}{2}z'_\nu\epsilon_P^2 & 0 \\ -\frac{15}{2}z_\nu\epsilon_P^2 & 15y_\nu\epsilon_G^2 e^{i\phi} & -\frac{1}{2}x_\nu\epsilon_G^2 \\ 0 & -\frac{1}{2}x_\nu\epsilon_G^2 & w_\nu \end{pmatrix} \tag{4}$$

for Dirac-type neutrino coupling. We will choose $n = 4$ in the following considerations. $\lambda_H$ is a universal coupling constant expected to be of order one. $\epsilon_G \equiv v_5/v_{10}$ and $\epsilon_P \equiv v_5/\bar{M}_P$ with $\bar{M}_P$, $v_{10}$ and $v_5$ being the VEVs for U(1)×Δ(48), SO(10) and SU(5) symmetry breakings respectively. $\phi$ is the physical CP phase[1] arising from the VEVs. The assumption of maximum CP violation implies that $\phi = \pi/2$. $x_f$, $y_f$, $z_f$, and $w_f$ ($f = u, d, e, \nu$) are the Clebsch factors of SO(10) determined by the directions of symmetry breaking of the adjoints **45**'s. The following three directions have been chosen for symmetry breaking, namely [2]

$$\begin{aligned} &< A_X >= v_{10}\ diag.(2,\ 2,\ 2,\ 2,\ 2) \otimes \tau_2, \\ &< A_z >= v_5\ diag.(-\frac{2}{3},\ -\frac{2}{3},\ -\frac{2}{3},\ -2,\ -2) \otimes \tau_2, \\ &< A_u >= v_5\ diag.(\frac{2}{3},\ \frac{2}{3},\ \frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}) \otimes \tau_2 \end{aligned} \tag{5}$$

Their corresponding U(1) hypercharges are given in Table 1.

---

[1] We have rotated away other possible phases by a phase redefinition of the fermion fields.

[2] In comparison with a direction in the Model I: $< A_z >= v_5\ diag.(\frac{2}{3},\ \frac{2}{3},\ \frac{2}{3},\ -2,\ -2) \otimes \tau_2$.

**Table 1.** U(1) Hypercharge Quantum Number

| | 'X' | 'u' | 'z' | B-L | $T_{3R}$ |
|---|---|---|---|---|---|
| $q$ | 1 | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | 0 |
| $u^c$ | 1 | 0 | $\frac{5}{3}$ | -$\frac{1}{3}$ | $\frac{1}{2}$ |
| $d^c$ | -3 | -$\frac{2}{3}$ | -$\frac{7}{3}$ | -$\frac{1}{3}$ | -$\frac{1}{2}$ |
| $l$ | - 3 | -1 | -1 | -1 | 0 |
| $e^c$ | 1 | $\frac{2}{3}$ | -1 | 1 | -$\frac{1}{2}$ |
| $\nu^c$ | 5 | $\frac{4}{3}$ | 3 | 1 | $\frac{1}{2}$ |

The Clebsch factors associated with the symmetry breaking directions can be easily read off from the U(1) hypercharges of the above table. The related effective operators obtained after the heavy fermion pairs are integrated out and decoupled are

$$
\begin{aligned}
W_{33} &= \lambda_H \;\; 16_3\, \eta_X \, \left(\frac{v_{10}}{A_X}\right)^{n+1} \, 10_1 \, \left(\frac{v_{10}}{A_X}\right)^{n+1} \, \eta_X \; 16_3 \\
W_{32} &= \lambda_H \epsilon_G^2 16_3 \; \eta_X \, \left(\frac{v_{10}}{A_X}\right)^{n+2} \left(\frac{A_z}{v_5}\right) 10_1 \left(\frac{A_z}{v_5}\right) \left(\frac{v_{10}}{A_X}\right)^{n+2} 16_2 \\
W_{22} &= \lambda_H \epsilon_G^2 16_2 \left(\frac{v_{10}}{A_X}\right)^{n+1} \left(\frac{A_u}{v_5}\right) 10_1 \left(\frac{A_u}{v_5}\right) \left(\frac{v_{10}}{A_X}\right)^{n+1} 16_2 e^{i\phi} \\
W_{12} &= \lambda_H \epsilon_P^2 \; 16_1 \; [\left(\frac{v_{10}}{A_X}\right)^{n-3} \; 10_1 \; \left(\frac{v_{10}}{A_X}\right)^{n-3} \\
&\quad + \left(\frac{v_{10}}{A_X}\right)^{n+1} \left(\frac{A_u}{v_5}\right) \; 10_1 \; \left(\frac{A_z}{v_5}\right) \left(\frac{v_{10}}{A_X}\right)^{n+2} ] \; 16_2
\end{aligned}
\tag{6}
$$

with $n = 4$ and $\phi = \pi/2$. The factor $\eta_X = 1/\sqrt{1 + 2(\frac{v_{10}}{A_X})^{2(n+1)}}$ in eq. (6) arises from mixing, and provides a factor of $1/\sqrt{3}$ for the up-type quark. It remains almost unity for the down-type quark and charged lepton as well as neutrino due to the suppression of large Clebsch factors in the second term of the square root. The relative phase (or sign) between the two terms in the operator $W_{12}$ has been fixed. The resulting Clebsch factors are

$$
\begin{aligned}
& w_u = w_d = w_e = w_\nu = 1, \\
& x_u = 5/9, \;\; x_d = 7/27, \;\; x_e = -1/3, \;\; x_\nu = 1/5, \\
& y_u = 0, \;\; y_d = y_e/3 = 2/27, \;\; y_\nu = 4/45,
\end{aligned}
\tag{7}
$$

$$z_u = 1,\ z_d = z_e = -27,\ z_\nu = -15^3 = -3375,$$
$$z'_u = 1 - 5/9 = 4/9,\ z'_d = z_d + 7/729 \simeq z_d,$$
$$z'_e = z_e - 1/81 \simeq z_e,\ z'_\nu = z_\nu + 1/15^3 \simeq z_\nu.$$

In obtaining the $\Gamma_f^G$ matrices, some small terms arising from mixings between the chiral fermion $16_i$ and the heavy fermion pairs $\psi_j(\bar{\psi}_j)$ are neglected. They are expected to change the numerical results no more than a few percent for the up-type quark mass matrix and are negligible for the down-type quark and lepton mass matrices due to the strong suppression of the Clebsch factors. This set of effective operators which lead to the above given Yukawa coupling matrices $\Gamma_f^G$ is quite unique for a successful prediction on fermion masses and mixings. A general superpotential leading to the above effective operators will be given in section 6. We would like to point out that unlike many other models in which $W_{33}$ is assumed to be a renormalizable interaction before symmetry breaking, the Yukawa couplings of all the quarks and leptons (both heavy and light) in both Model II and Model I are generated at the GUT scale after the breakdown of the family group and SO(10). Therefore, initial conditions for renormalization group (RG) evolution will be set at the GUT scale for all the quark and lepton Yukawa couplings. The hierarchy among the three families is described by the two ratios $\epsilon_G = v_5/v_{10}$ and $\epsilon_P = v_5/\bar{M}_P$. The mass splittings between the quarks and leptons as well as between the up and down quarks are determined by the Clebsch factors of SO(10). From the GUT scale down to low energies, Renormalization Group (RG) evolution has been taken into account. The top-bottom splitting in the present model is mainly attributed to the Clebsch factor $1/3^n$ with $n = 4$ rather than the large value of $\tan\beta$ caused by the hierarchy of the VEVs $v_1$ and $v_2$ of the two light Higgs doublets.

An adjoint **45** $A_X$ and a 16-dimensional representation Higgs field $\Phi$ $(\bar{\Phi})$ are needed for breaking SO(10) down to SU(5). Adjoint **45** $A_z$ and $A_u$ are needed to break SU(5) further down to the standard model SU(3)$_c\times$ SU$_L$(2)$\times$ U(1)$_Y$.

## III. PREDICTIONS

From the Yukawa coupling matrices given above with $n = 4$ and $\phi = \pi/2$, the 13 parameters in the SM can be determined by only four parameters: a universal coupling constant $\lambda_H$ and three ratios of the VEVs: $\epsilon_G = v_5/v_{10}$, $\epsilon_P = v_5/\bar{M}_P$ and $\tan\beta = v_2/v_1$. In obtaining physical masses and mixings, renormalization group (RG) effects has been taken into consideration. The result at the low energy obtained by scaling down from the GUT scale will depend on the strong coupling constant $\alpha_s$. From low-energy measurements [26] and lattice calculations [27], $\alpha_s$ at the scale $M_Z$, has value around $\alpha_s(M_Z) = 0.113$, which was also found to be consistent with a recent global fit [28] to the LEP data. This value might be reached in nonminimal SUSY GUT models through large threshold effects. As our focus here is on the fermion masses and mixings, we shall not discuss it in this paper. In the present consideration, we take $\alpha_s(M_Z) \simeq 0.113$. The prediction on fermion masses and mixings thus obtained is found to be remarkable. Our numerical predictions are given in table 2b with four input parameters given in table 2a:

**Table 2a.** Input parameters and their values.

| $m_e[MeV]$ | $m_\mu[MeV]$ | $m_\tau[GeV]$ | $m_b(m_b)[GeV]$ |
|---|---|---|---|
| 0.511 | 105.66 | 1.777 | 4.25 |

**Table 2b.** Output parameters and their predicted values with input parameters given in table 2a and $\alpha_s(M_Z) = 0.113$.

| Output parameters | Output values | Data [23–25] | Output para. | Output values |
|---|---|---|---|---|
| $M_t$ [GeV] | 182 | $180 \pm 15$ | $J_{CP} = A^2\lambda^6\eta$ | $2.68 \times 10^{-5}$ |
| $m_c(m_c)$ [GeV] | 1.27 | $1.27 \pm 0.05$ | $\alpha$ | $86.28^\circ$ |
| $m_u(1\text{GeV})$ [MeV] | 4.31 | $4.75 \pm 1.65$ | $\beta$ | $22.11^\circ$ |
| $m_s(1\text{GeV})$ [MeV] | 156.5 | $165 \pm 65$ | $\gamma$ | $71.61^\circ$ |
| $m_d(1\text{GeV})$ [MeV] | 6.26 | $8.5 \pm 3.0$ | $m_{\nu_\tau}$ [eV] | 2.451536 |
| $\lvert V_{us}\rvert = \lambda$ | 0.22 | $0.221 \pm 0.003$ | $m_{\nu_\mu}$ [eV] | 2.448464 |
| $\frac{\lvert V_{ub}\rvert}{\lvert V_{cb}\rvert} = \lambda\sqrt{\rho^2+\eta^2}$ | 0.083 | $0.08 \pm 0.03$ | $m_{\nu_e}$ [eV] | $1.27 \times 10^{-3}$ |
| $\frac{\lvert V_{td}\rvert}{\lvert V_{ts}\rvert} = \lambda\sqrt{(1-\rho)^2+\eta^2}$ | 0.209 | $0.24 \pm 0.11$ | $m_{\nu_s}$ [eV] | $2.8 \times 10^{-3}$ |
| $\lvert V_{cb}\rvert = A\lambda^2$ | 0.0393 | $0.039 \pm 0.005$ | $\lvert V_{\nu_\mu e}\rvert$ | -0.049 |
| $\lambda_t^G$ | 1.30 | - | $\lvert V_{\nu_e \tau}\rvert$ | 0.000 |
| $\tan\beta = v_2/v_1$ | 2.33 | - | $\lvert V_{\nu_\tau e}\rvert$ | -0.049 |
| $\epsilon_G = v_5/v_{10}$ | $2.987 \times 10^{-1}$ | - | $\lvert V_{\nu_\mu \tau}\rvert$ | -0.707 |
| $\epsilon_P = v_5/\bar{M}_P$ | $1.011 \times 10^{-2}$ | - | $\lvert V_{\nu_e s}\rvert$ | $3.8 \times 10^{-2}$ |
| $B_K$ | 0.90 | $0.82 \pm 0.10$ | $M_{N_1}$ [GeV] | $\sim 333$ |
| $f_B\sqrt{B}$ [MeV] | 207 | $200 \pm 70$ | $M_{N_2}$ [GeV] | $1.63 \times 10^6$ |
| $\text{Re}(\varepsilon'/\varepsilon)$ | $(1.4 \pm 1.0) \times 10^{-3}$ | $(1.5 \pm 0.8) \times 10^{-3}$ | $M_{N_3}$ [GeV] | 333 |

The predictions on the quark masses and mixings as well as CP-violating effects presented above agree remarkably with those extracted from various experimental data. Especially, there are four predictions on $|V_{us}|$, $|V_{ub}/V_{cb}|$, $|V_{td}/V_{ts}|$ and $m_d/m_s$ which are independent of the RG scaling (see eqs. (41)-(44) below).

Let us now analyze in detail the above predictions. To a good approximation, the up-type and down-type quark Yukawa coupling matrices can be diagonalized in the form

$$V_d = \begin{pmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-i\phi} & 0 & 0 \\ 0 & c_d & s_d \\ 0 & -s_d & c_d \end{pmatrix} \tag{8}$$

$$V_u = \begin{pmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-i\phi} & 0 & 0 \\ 0 & c_u & s_u \\ 0 & -s_u & c_u \end{pmatrix} \tag{9}$$

The CKM matrix at the GUT scale is then given by $V_{CKM} = V_u V_d^\dagger$. Where $s_i \equiv \sin(\theta_i)$ and $c_i \equiv \cos(\theta_i)$ $(i = 1, 2, u, d)$. For $\phi = \pi/2$, the angles $\theta_i$ at the GUT scale are given by

$$\tan(\theta_1) \simeq -\frac{z_d}{2y_d}\frac{\epsilon_P^2}{\epsilon_G^2}, \qquad \tan(\theta_2) \simeq \frac{2w_u z_u}{x_u^2}\frac{\epsilon_P^2}{\epsilon_G^4}, \tag{10}$$

$$\tan(\theta_d) \simeq \frac{x_d}{2w_d}\epsilon_G^2, \qquad \tan(\theta_u) \simeq \frac{\sqrt{3}}{2}\frac{x_u}{w_u}\epsilon_G^2. \tag{11}$$

and the Yukawa eigenvalues at the GUT scale are found to be

$$\frac{\lambda_u^G}{\lambda_c^G} = \frac{4w_u^2 z_u z_u'}{x_u^4 \epsilon_G^4}\frac{\epsilon_P^4}{\epsilon_G^4}, \qquad \frac{\lambda_c^G}{\lambda_t^G} = \frac{3}{4}\frac{x_u^2}{w_u^2}\epsilon_G^4, \tag{12}$$

$$\frac{\lambda_d^G}{\lambda_s^G}\left(1 - \frac{\lambda_d^G}{\lambda_s^G}\right)^{-2} = \frac{z_d^2}{4y_d^2}\frac{\epsilon_P^4}{\epsilon_G^4}, \qquad \frac{\lambda_s^G}{\lambda_b^G} = 3\frac{y_d}{w_d}\epsilon_G^2, \tag{13}$$

$$\frac{\lambda_e^G}{\lambda_\mu^G}\left(1 - \frac{\lambda_e^G}{\lambda_\mu^G}\right)^{-2} = \frac{z_e^2}{4y_e^2}\frac{\epsilon_P^4}{\epsilon_G^4}, \qquad \frac{\lambda_\mu^G}{\lambda_\tau^G} = 3\frac{y_e}{w_e}\epsilon_G^2. \tag{14}$$

Using the eigenvalues and angles of these Yukawa matrices , one can easily find the following ten relations among fermion masses and CKM matrix elements at the GUT scale

$$\left(\frac{m_b}{m_\tau}\right)_G = 1, \tag{15}$$

$$\left(\frac{m_s}{m_\mu}\right)_G = \frac{1}{3}, \qquad or \qquad \left(\frac{m_s}{m_b}\right)_G = \frac{1}{3}\left(\frac{m_\mu}{m_\tau}\right)_G \tag{16}$$

$$\left(\frac{m_d}{m_s}\right)_G \left(1 - (\frac{m_d}{m_s})_G\right)^{-2} = 9\left(\frac{m_e}{m_\mu}\right)_G \left(1 - \left(\frac{m_e}{m_\mu}\right)_G\right)^{-2}, \tag{17}$$

$$\left(\frac{m_t}{m_\tau}\right)_G = 81\ \tan\beta\ , \tag{18}$$

$$\left(\frac{m_c}{m_t}\right)_G = \frac{25}{48}\left(\frac{m_\mu}{m_\tau}\right)_G^2\ , \tag{19}$$

$$\left(\frac{m_u}{m_c}\right)_G = \frac{4}{9}\left(\frac{4}{15}\right)^4 \left(\frac{m_e m_\tau^2}{m_\mu^3}\right)_G\ , \tag{20}$$

$$|\frac{V_{ub}}{V_{cb}}|_G = \tan(\theta_2) = \left(\frac{4}{15}\right)^2 \left(\frac{m_\tau}{m_\mu}\right)_G \sqrt{\left(\frac{m_e}{m_\mu}\right)_G}\ , \tag{21}$$

$$|\frac{V_{td}}{V_{ts}}|_G = \tan(\theta_1) = 3\sqrt{\left(\frac{m_e}{m_\mu}\right)_G}\ , \tag{22}$$

$$|V_{us}|_G = c_1 c_2 \sqrt{\tan^2(\theta_1)+\tan^2(\theta_2)} = 3\sqrt{\left(\frac{m_e}{m_\mu}\right)_G}\left(\frac{1+(\frac{16}{675}(\frac{m_\tau}{m_\mu})_G)^2}{1+9(\frac{m_e}{m_\mu})_G}\right)^{1/2} , \tag{23}$$

$$|V_{cb}|_G = c_2 c_d c_u(\tan(\theta_u)-\tan(\theta_d)) = \frac{15\sqrt{3}-7}{15\sqrt{3}}\frac{5}{4\sqrt{3}}\left(\frac{m_\mu}{m_\tau}\right)_G . \tag{24}$$

The Clebsch factors in eq. (7) appeared as those miraculus numbers in the above relations. The index 'G' refers throughout to quantities at the GUT scale. The first two relations are well-known in the Georgi-Jarlskog texture. The physical fermion masses and mixing angles are related to the above Yukawa eigenvalues and angles through the renormalization group (RG) equations [29]. As most Yukawa couplings in the present model are much smaller than the top quark Yukawa coupling $\lambda_t^G \sim 1$. In a good approximation, we will only keep top quark Yukawa coupling terms in the RG equations and neglect all other Yukawa coupling terms in the RG equations. The RG evolution will be described by three kinds of scaling factors. Two of them ($\eta_F$ and $R_t$ ) arise from running the Yukawa parameters from the GUT scale down to the SUSY breaking scale $M_S$ which is chosen to be close to the top quark mass, i.e., $M_S \simeq m_t \simeq 170$ GeV, and are defined as

$$m_t(M_S) = \eta_U(M_S)\ \lambda_t^G\ R_t^{-6}\ \frac{v}{\sqrt{2}}\sin\beta , \tag{25}$$

$$m_b(M_S) = \eta_D(M_S)\ \lambda_b^G\ R_t^{-1}\ \frac{v}{\sqrt{2}}\cos\beta , \tag{26}$$

$$m_i(M_S) = \eta_U(M_S)\ \lambda_i^G\ R_t^{-3}\ \frac{v}{\sqrt{2}}\sin\beta , \qquad i=u,c, \tag{27}$$

$$m_i(M_S) = \eta_D(M_S)\lambda_i^G\frac{v}{\sqrt{2}}\cos\beta , \qquad i=d,s, \tag{28}$$

$$m_i(M_S) = \eta_E(M_S)\lambda_i^G\frac{v}{\sqrt{2}}\cos\beta , \qquad i=e,\mu,\tau, \tag{29}$$

$$\lambda_i(M_S) = \eta_N(M_S)\ \lambda_i^G\ R_t^{-3} , \qquad i=\nu_e,\nu_\mu,\nu_\tau . \tag{30}$$

with $v = 246$ GeV. $\eta_F(M_S)$ and $R_t$ are given by

$$\eta_F(M_S) = \prod_{i=1}^{3}\left(\frac{\alpha_i(M_G)}{\alpha_i(M_S)}\right)^{c_i^F/2b_i} , \qquad F=U,D,E,N \tag{31}$$

$$R_t^{-1} = e^{-\int_{\ln M_S}^{\ln M_G}(\frac{\lambda_t(t)}{4\pi})^2 dt} = (1+(\lambda_t^G)^2 K_t)^{-1/12} = \left(1-\frac{\lambda_t^2(M_S)}{\lambda_f^2}\right)^{1/12} \tag{32}$$

with $c_i^U = (\frac{13}{15},3,\frac{16}{3})$, $c_i^D = (\frac{7}{15},3,\frac{16}{3})$ , $c_i^E = (\frac{27}{15},3,0)$, $c_i^N = (\frac{9}{25},3,0)$, and $b_i = (\frac{33}{5},1,-3)$, where $\lambda_f$ is the fixed point value of $\lambda_t$ and is given by

$$\lambda_f = \frac{2\pi\eta_U^2}{\sqrt{3I(M_S)}}, \qquad I(M_S) = \int_{\ln M_S}^{\ln M_G} \eta_U^2(t)dt \tag{33}$$

The factor $K_t$ is related to the fixed point value via $K_t = \eta_U^2/\lambda_f^2 = \frac{3I(M_S)}{4\pi^2}$ . The numerical value for $I$ taken from Ref. [30] is 113.8 for $M_S \simeq m_t = 170$GeV. $\lambda_f$ cannot be equal to $\lambda_t(M_S)$ exactly , since that would correspond to infinite $\lambda_t^G$ , and lead to the so called Landau pole problem at the GUT scale. Other RG scaling factors are derived by running Yukawa couplings below $M_S$

$$m_i(m_i) = \eta_i \; m_i(M_S), \qquad i = c, b, \tag{34}$$

$$m_i(1GeV) = \eta_i \; m_i(M_S), \qquad i = u, d, s \tag{35}$$

where $\eta_i$ are the renormalization factors. The physical top quark mass is given by

$$M_t = m_t(m_t)\left(1 + \frac{4}{3}\frac{\alpha_s(m_t)}{\pi}\right) \tag{36}$$

In numerical calculations, we take $\alpha^{-1}(M_Z) = 127.9$, $s^2(M_Z) = 0.2319$, $M_Z = 91.187$ GeV and use the gauge couplings at $M_G \sim 2 \times 10^{16}$ GeV at GUT scale and that of $\alpha_1$ and $\alpha_2$ at $M_S \simeq m_t \simeq 170$ GeV

$$\alpha_1^{-1}(m_t) = \alpha_1^{-1}(M_Z) + \frac{53}{30\pi}\ln\frac{M_Z}{m_t} = 58.59, \tag{37}$$

$$\alpha_2^{-1}(m_t) = \alpha_2^{-1}(M_Z) - \frac{11}{6\pi}\ln\frac{M_Z}{m_t} = 30.02, \tag{38}$$

$$\alpha_1^{-1}(M_G) = \alpha_2^{-1}(M_G) = \alpha_3^{-1}(M_G) \simeq 24 \tag{39}$$

we keep $\alpha_3(M_Z)$ as a free parameter in this note. The precise prediction on $\alpha_3(M_Z)$ concerns GUT and SUSY threshold corrections. We shall not discuss it here since our focus in this note is the fermion masses and mixings. Including the three-loop QCD and one-loop QED contributions, the following values of $\eta_i$ will be used in numerical calculations.

**Table 3.** Values of $\eta_i$ and $\eta_F$ as a function of the strong coupling $\alpha_s(M_Z)$

| $\alpha_s(M_Z)$ | 0.110 | 0.113 | 0.115 | 0.117 | 0.120 |
|---|---|---|---|---|---|
| $\eta_{u,d,s}$ | 2.08 | 2.20 | 2.26 | 2.36 | 2.50 |
| $\eta_c$ | 1.90 | 2.00 | 2.05 | 2.12 | 2.25 |
| $\eta_b$ | 1.46 | 1.49 | 1.50 | 1.52 | 1.55 |
| $\eta_{e,\mu,\tau}$ | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 |
| $\eta_U$ | 3.26 | 3.33 | 3.38 | 3.44 | 3.50 |
| $\eta_D/\eta_E \equiv \eta_{D/E}$ | 2.01 | 2.06 | 2.09 | 2.12 | 2.16 |
| $\eta_E$ | 1.58 | 1.58 | 1.58 | 1.58 | 1.58 |
| $\eta_N$ | 1.41 | 1.41 | 1.41 | 1.41 | 1.41 |

It is interesting to note that the mass ratios of the charged leptons are almost independent of the RG scaling factors since $\eta_e = \eta_\mu = \eta_\tau$ (up to an accuracy $O(10^{-3})$), namely

$$\frac{m_e}{m_\mu} = \left(\frac{m_e}{m_\mu}\right)_G, \qquad \frac{m_\mu}{m_\tau} = \left(\frac{m_\mu}{m_\tau}\right)_G \tag{40}$$

which is different from the models with large $\tan\beta$. In the present model the $\tau$ lepton Yukawa coupling is small. It is easily seen that four relations represented by eqs. (21)-(23) and (17) hold at low energies. Using the known lepton masses $m_e = 0.511$ MeV, $m_\mu = 105.66$ MeV, and $m_\tau = 1.777$ GeV, we obtain four important RG scaling-independent predictions:

$$|V_{us}| = |V_{us}|_G = \lambda \simeq 3\sqrt{\frac{m_e}{m_\mu}}\left(\frac{1+(\frac{16}{675}\frac{m_\tau}{m_\mu})^2}{1+9\frac{m_e}{m_\mu}}\right)^{1/2} = 0.22, \tag{41}$$

$$|\frac{V_{ub}}{V_{cb}}| = |\frac{V_{ub}}{V_{cb}}|_G = \lambda\sqrt{\rho^2+\eta^2} \simeq \left(\frac{4}{15}\right)^2 \frac{m_\tau}{m_\mu}\sqrt{\frac{m_e}{m_\mu}} = 0.083, \tag{42}$$

$$|\frac{V_{td}}{V_{ts}}| = |\frac{V_{td}}{V_{ts}}|_G = \lambda\sqrt{(1-\rho)^2+\eta^2} \simeq 3\sqrt{\frac{m_e}{m_\mu}} = 0.209, \tag{43}$$

$$\frac{m_d}{m_s}\left(1-\frac{m_d}{m_s}\right)^{-2} = 9\frac{m_e}{m_\mu}\left(1-\frac{m_e}{m_\mu}\right)^{-2}, \qquad i.e., \qquad \frac{m_d}{m_s} = 0.040 \tag{44}$$

and six RG scaling-dependent predictions:

$$|V_{cb}| = |V_{cb}|_G R_t = A\lambda^2 = \frac{15\sqrt{3}-7}{15\sqrt{3}}\frac{5}{4\sqrt{3}}\frac{m_\mu}{m_\tau}R_t = 0.0391\ \left(\frac{0.80}{R_t^{-1}}\right), \tag{45}$$

$$m_s(1GeV) = \frac{1}{3}m_\mu\frac{\eta_s}{\eta_\mu}\eta_{D/E} = 159.53\ \left(\frac{\eta_s}{2.2}\right)\left(\frac{\eta_{D/E}}{2.1}\right)\ MeV, \tag{46}$$

$$m_b(m_b) = m_\tau \frac{\eta_b}{\eta_\tau}\eta_{D/E} R_t^{-1} = 4.25 \ \left(\frac{\eta_b}{1.49}\right)\left(\frac{\eta_{D/E}}{2.04}\right)\left(\frac{R_t^{-1}}{0.80}\right) \ GeV \ , \tag{47}$$

$$m_u(1GeV) = \frac{5}{3}(\frac{4}{45})^3 \frac{m_e}{m_\mu}\eta_u R_t^3 m_t = 4.23 \ \left(\frac{\eta_u}{2.2}\right)\left(\frac{0.80}{R_t^{-1}}\right)^3 \left(\frac{m_t(m_t)}{174GeV}\right) \ MeV \ , \tag{48}$$

$$m_c(m_c) = \frac{25}{48}(\frac{m_\mu}{m_\tau})^2 \eta_c R_t^3 m_t = 1.25 \ \left(\frac{\eta_c}{2.0}\right)\left(\frac{0.80}{R_t^{-1}}\right)^3 \left(\frac{m_t(m_t)}{174GeV}\right) \ GeV, \tag{49}$$

$$m_t(m_t) = \frac{\eta_U}{\sqrt{K_t}}\sqrt{1-R_t^{-12}}\frac{v}{\sqrt{2}}\sin\beta = 174.9 \ \left(\frac{\sin\beta}{0.92}\right)\left(\frac{\eta_U}{3.33}\right)\left(\sqrt{\frac{8.65}{K_t}}\right)\left(\frac{\sqrt{1-R_t^{-12}}}{0.965}\right) \ GeV \tag{50}$$

We have used the fixed point property for the top quark mass. These predictions depend on two parameters $R_t$ and $\sin\beta$ (or $\lambda_t^G$ and $\tan\beta$). In general, the present model contains four parameters: $\epsilon_G = v_5/v_{10}$, $\epsilon_P = v_5/\bar{M}_P$, $\tan\beta = v_2/v_1$, and $\lambda_t^G = 81\lambda_b^G = 81\lambda_\tau^G = \frac{2}{3}\lambda_H$. It is not difficult to notice that $\epsilon_G$ and $\epsilon_P$ are determined solely by the Clebsch factors and mass ratios of the charged leptons

$$\epsilon_G = \frac{v_5}{v_{10}} = \sqrt{\frac{m_\mu}{m_\tau}\frac{\eta_\tau}{\eta_\mu}\frac{w_e}{3y_e}} = 2.987 \times 10^{-1}, \tag{51}$$

$$\epsilon_P = \frac{v_5}{\bar{M}_P} = \left(\frac{4}{9}\frac{m_e m_\mu}{m_\tau^2}\frac{\eta_\tau^2}{\eta_e\eta_\mu}\frac{w_e^2}{z_e^2}\right)^{1/4} = 1.011 \times 10^{-2}. \tag{52}$$

The coupling $\lambda_t^G$ (or $R_t$) can be determined by the mass ratio of the bottom quark and $\tau$ lepton

$$\lambda_t^G = \frac{1}{\sqrt{K_t}}\frac{\sqrt{1-R_t^{-12}}}{R_t^{-6}} = 1.25 \ \zeta_t, \tag{53}$$

$$\zeta_t \equiv \left(\sqrt{\frac{8.65}{K_t}}\right)\left(\frac{0.80}{R_t^{-1}}\right)^6 \left(\frac{\sqrt{1-R_t^{-12}}}{0.965}\right), \tag{54}$$

$$R_t^{-1} = \frac{m_b}{m_\tau}\frac{\eta_\tau}{\eta_b}\frac{1}{\eta_{D/E}} = 0.80 \ \left(\frac{m_b(m_b)}{4.25GeV}\right)\left(\frac{1.49}{\eta_b}\right)\left(\frac{2.04}{\eta_{D/E}}\right). \tag{55}$$

$\tan\beta$ is fixed by the $\tau$ lepton mass

$$\begin{aligned}
\cos\beta &= \frac{m_\tau\sqrt{2}}{\eta_E\eta_\tau v\lambda_\tau^G} = \left(\frac{0.41}{\zeta_t}\right)\left(\frac{3^n}{81}\right), \\
\sin\beta &= \sqrt{1-(\frac{0.41}{\zeta_t}\frac{3^n}{81})^2} = 0.912 \ \left(\frac{\sqrt{1-(\frac{0.41}{\zeta_t}\frac{3^n}{81})^2}}{0.912}\right), \\
\tan\beta &= 2.225 \ \left(\frac{81}{3^n}\right)\left(\frac{\sqrt{\zeta_t^2-(0.41)^2(3^n/81)^2}}{0.912}\right).
\end{aligned} \tag{56}$$

With these considerations, the top quark mass is given by

$$m_t(m_t) = 173.4 \ \left(\frac{\eta_U}{3.33}\right)\left(\sqrt{\frac{8.65}{K_t}}\right)\left(\frac{\sqrt{1-R_t^{-12}}}{0.965}\right)\left(\frac{\sqrt{1-(0.41/\zeta_t)^2}}{0.912}\right) \ GeV \tag{57}$$

Given $\epsilon_G$ and $\epsilon_P$ as well $\lambda_t^G$, the Yukawa coupling matrices of the fermions at the GUT scale are then known. It is of interest to expand the above fermion Yukawa coupling matrices $\Gamma_f^G$ in terms of the parameter $\lambda = 0.22$ (the Cabbibo angle), as Wolfenstein [31] did for the CKM mixing matrix.

$$\Gamma_u^G = 1.25\zeta_t \begin{pmatrix} 0 & 0.60\lambda^6 & 0 \\ 1.35\lambda^6 & 0 & -0.89\lambda^2 \\ 0 & -0.89\lambda^2 & 1 \end{pmatrix}, \tag{58}$$

$$\Gamma_d^G = -\frac{1.25\zeta_t}{81} \begin{pmatrix} 0 & 1.77\lambda^4 & 0 \\ 1.77\lambda^4 & 0.41\lambda^2 e^{i\frac{\pi}{2}} & -1.09\lambda^3 \\ 0 & -1.09\lambda^3 & 1 \end{pmatrix}, \tag{59}$$

$$\Gamma_e^G = -\frac{1.25\zeta_t}{81} \begin{pmatrix} 0 & 1.77\lambda^4 & 0 \\ 1.77\lambda^4 & 1.23\lambda^2 e^{i\frac{\pi}{2}} & 1.40\lambda^2 \\ 0 & 1.40\lambda^2 & 1 \end{pmatrix}, \tag{60}$$

$$\Gamma_\nu^G = -\left(\frac{1.25\zeta_t}{81}\right)\left(\frac{2.581}{5^5}\right) \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0.95\lambda^2 e^{i\frac{\pi}{2}} & 1.472\lambda^4 \\ 0 & 1.472\lambda^4 & 1.757\lambda \end{pmatrix} \tag{61}$$

Using the CKM parameters and quark masses predicted in the present model, the bag parameter $B_K$ can be extracted from the indirect CP-violating parameter $|\varepsilon_K| = 2.6 \times 10^{-3}$ in $K^0$-$\bar{K}^0$ system via

$$B_K = 0.90 \ \left(\frac{0.57}{\eta_2}\right)\left(\frac{|\varepsilon_K|}{2.6\times 10^{-3}}\right)\left(\frac{0.138 y_t^{1.55}}{A^4(1-\rho)\eta}\right)\left(\frac{1.41}{1+\frac{0.246 y_t^{1.34}}{A^2(1-\rho)}}\right) \tag{62}$$

The B-meson decay constant can also be obtained from fitting the $B^0$-$\bar{B}^0$ mixing

$$f_B\sqrt{B} = 207 \ \left(\sqrt{\frac{0.55}{\eta_B}}\right)\left(\frac{\Delta M_{B_d}(ps^{-1})}{0.465}\right)\left(\frac{0.77 y_t^{0.76}}{A\sqrt{(1-\rho)^2+\eta^2}}\right) \ MeV \tag{63}$$

with $y_t = 175GeV/m_t(m_t)$ and $\eta_2$ and $\eta_B$ being the QCD corrections [32]. Note that we did not consider the possible contributions to $\varepsilon_K$ and $\Delta M_{B_d}$ from box diagrams through exchanges of superparticles. To have a complete analysis, these contributions should be included in a more detailed consideration in the future. The parameter $B_K$ was estimated ranging from 1/3 to 1 based on various approaches. Recent analysis using the lattice methods [33,34] gives $B_K = 0.82 \pm 0.1$ . There are also various calculations on the parameter $f_{B_d}$. From the recent lattice analyses [33,35], $f_{B_d} = (200 \pm 40)$ MeV, $B_{B_d} = 1.0 \pm 0.2$. QCD sum rule calculations [36] also gave a compatible result. An interesting upper bound [37] $f_B\sqrt{B} <$ 213MeV for $m_c = 1.4$GeV and $m_b = 4.6$ GeV or $f_B\sqrt{B} <$ 263MeV for $m_c = 1.5$GeV and $m_b = 5.0$ GeV has been obtained by relating the hadronic mixing matrix element, $\Gamma_{12}$, to the decay rate of the bottom quark.

The direct CP-violating parameter $\mathrm{Re}(\varepsilon'/\varepsilon)$ in the K-system has been estimated by the standard method. The uncertanties mainly arise from the hadronic matrix elements [38]. We have included the next-to-leading order contributions from the chiral-loop [39–41] and the next-to-leading order perturbative contributions [42,43] to the Wilson coefficients together with a consistent analysis of the $\Delta I = 1/2$ rule. Experimental results on $\mathrm{Re}(\varepsilon'/\varepsilon)$ is inconclusive. The NA31 collaboration at CERN reported a value $\mathrm{Re}(\varepsilon'/\varepsilon) = (2.3 \pm 0.7) \cdot 10^{-3}$ [44] which clearly indicates direct CP violation, while the value given by E731 at Fermilab, $\mathrm{Re}(\varepsilon'/\varepsilon) = (0.74 \pm 0.59) \cdot 10^{-3}$ [45] is compatible with superweak theories [46] in which $\varepsilon'/\varepsilon = 0$. The average value quoted in [25] is $\mathrm{Re}(\varepsilon'/\varepsilon) = (1.5 \pm 0.8) \cdot 10^{-3}$.

For predicting physical observables, it is better to use $J_{CP}$ , the rephase-invariant CP-violating quantity, together with $\alpha$, $\beta$ and $\gamma$ , the three angles of the unitarity triangle of a three-family CKM matrix

$$\alpha = arg. \left( -\frac{V_{td}V_{tb}^*}{V_{ud}V_{ub}^*} \right), \qquad \beta = arg. \left( -\frac{V_{cd}V_{cb}^*}{V_{td}V_{tb}^*} \right), \qquad \gamma = arg. \left( -\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*} \right) \tag{64}$$

where $\sin 2\alpha$, $\sin 2\beta$ and $\sin 2\gamma$ can in principle be measured in $B^0/\bar{B}^0 \to \pi^+\pi^-$ [47], $J/\psi K_S$ [48] and $B^- \to K^- D$ [49], respectively. $|V_{us}|$ has been extracted with good accuracy from $K \to \pi e \nu$ and hyperon decays [25]. $|V_{cb}|$ can be determined from both exclusive and inclusive

semileptonic $B$ decays with values given by

$$|V_{cb}| = \begin{cases} 0.039 \pm 0.001\ (exp.) \pm 0.005\ (theor.); & \text{measurements at } \Upsilon(4s), \\ 0.042 \pm 0.002\ (exp.) \pm 0.005\ (theor.) & \text{measurements at } Z^0 \end{cases} \tag{65}$$

from inclusive semileptonic $B$ decays [50] and

$$|V_{cb}| = \begin{cases} 0.0407 \pm 0.0027\ (exp.) \pm 0.0016\ (theor.); & [51] \\ 0.0388 \pm 0.0019\ (exp.) \pm 0.0017\ (theor.) & [52] \end{cases} \tag{66}$$

from the exclusive semileptonic B decays. The data from the exclusive channels is taken from the results by CLEO, ALEPH, ARGUS and DELPHI

$$|V_{cb}|\mathcal{F}(1) = \begin{cases} 0.0351 \pm 0.0019 \pm 0.0020; & \text{[CLEO]} \\ 0.0314 \pm 0.0023 \pm 0.0025; & \text{[ALEPH]} \\ 0.0388 \pm 0.0043 \pm 0.0025; & \text{[ARGUS]} \\ 0.0374 \pm 0.0021 \pm 0.0034; & \text{[DELPHI]} \\ 0.0370 \pm 0.0025; \text{WEIGHTED AVERAGE in [51]} & \\ 0.0353 \pm 0.0018; \text{WEIGHTED AVERAGE in [52]} & \end{cases}$$

with the related Isgur-Wise function $\mathcal{F}(1)$ taking the value [51,53]

$$\mathcal{F}(1) = 0.91 \pm 0.04$$

The above values are also in good agreement with the value $|V_{cb}| = 0.037^{+0.003}_{-0.002}$ obtained from the exclusive decay $B \to D^* l \nu_l$ by using a dispersion relation approach [54].

Another CKM parameter $|V_{ub}/V_{cb}|$ is extracted from a study of the semileptonic $B$ decays near the end point region of the lepton spectrum. The present experimental measurements are compatible with

$$|\frac{V_{ub}}{V_{cb}}| = 0.08 \pm 0.01\ (exp.) \pm 0.02\ (theor.) \tag{67}$$

The CKM parameter $|V_{td}/V_{ts}|$ is constrained [52] by the indirect CP-violating parameter $|\varepsilon|$ in kaon decays and $B^0$-$\bar{B}^0$ mixing $x_d$. Large uncertainties of $|V_{td}/V_{ts}|$ are caused by the bag parameter $B_K$ and the leptonic $B$ decay constant $f_B$.

A detail analysis of neutrino masses and mixings will be presented in the next section. Before proceeding further, we would like to address the following points: Firstly, given $\alpha_s(M_Z)$ and $m_b(m_b)$, the value of $\tan\beta$ depends, as one sees from eq.(56), on the choice of the integer 'n' in an over all factor $1/3^n$, so do the masses of all the up-type quarks (see eqs. (48)-(50)). For $n > 4$, the value of $\tan\beta$ becomes too small, as a consequence, the resulting top quark mass will be below the present experimental lower bound, so do the masses of the up and charm quarks. In contrast, for $1 < n < 4$, the values of $\tan\beta$ will become larger, the resulting charm quark mass will be above the present upper bound and the top quark mass is very close to the present upper bound. Secondly, given $m_b(m_b)$ and integer 'n', all other quark masses increase with $\alpha_s(M_Z)$. This is because the RG scaling factors $\eta_i$ and $R_t$ increase with $\alpha_s(M_Z)$. When $\alpha_s(M_Z)$ is larger than 0.117 and n=4, either charm quark mass or bottom quark mass will be above the present upper bound. Finally, the symmetry breaking direction of the adjoint **45** $A_z$ or the Clebsch factor $x_u$ is strongly restricted by both $|V_{ub}|/|V_{cb}|$ and charm quark mass $m_c(m_c)$. From these considerations, we conclude that the best choice of n will be 4 for small $\tan\beta$ and the value of $\alpha_s$ should around $\alpha_s(M_Z) \simeq 0.113$ , which can be seen from table 2b.

## IV. NEUTRINO MASSES AND MIXINGS

Neutrino masses and mixings , if they exist, are very important in astrophysics and crucial for model building. Many unification theories predict a see-saw type mass [55] $m_{\nu_i} \sim m_{u_i}^2/M_N$ with $u_i = u, c, t$ being up-type quarks. For $M_N \simeq (10^{-3} \sim 10^{-4})M_{GUT} \simeq 10^{12} - 10^{13}$ GeV, one has

$$m_{\nu_e} < 10^{-7}eV, \qquad m_{\nu_\mu} \sim 10^{-3}eV, \qquad m_{\nu_\tau} \sim (3-21)eV \tag{68}$$

In this case solar neutrino anomalous could be explained by $\nu_e \to \nu_\mu$ oscillation, and the mass of $\nu_\tau$ is in the range relevant to hot dark matter. However, LSND events and atmospheric neutrino deficit can not be explained in this scenario.

By choosing Majorana type Yukawa coupling matrix differently, one can construct many models of neutrino mass matrix. As we have shown in the Model I that by choosing an appropriate texture structure with some diagonal zero elements in the right-handed Majorana mass matrix, one can explain the recent LSND events, atmospheric neutrino deficit and hot dark matter, however, the solar neutrino anomalous can only be explained by introducing a sterile neutrino. A similar consideration can be applied to the present model. The followng texture structure with zeros is found to be interesting for the present model

$$M_N^G = \lambda_H \frac{v_{10}}{\bar{M}_P} \epsilon_P^4 \epsilon_G^2 v_{10} \begin{pmatrix} 0 & 0 & \frac{1}{2} z_N \epsilon_P^2 e^{i(\delta_\nu + \phi_3)} \\ 0 & y_N e^{2i\phi_2} & 0 \\ \frac{1}{2} z_N \epsilon_P^2 e^{i(\delta_\nu + \phi_3)} & 0 & w_N \epsilon_P^4 e^{2i\phi_3} \end{pmatrix} \tag{69}$$

The corresponding effective operators are given by

$$\begin{aligned} W_{22}^N &= \lambda_H v_{10} \frac{v_{10}}{\bar{M}_P} \epsilon_P^4 16_2 (\frac{A_z}{A_X})(\frac{\bar{\Phi}}{v_{10}})(\frac{\bar{\Phi}}{v_{10}})(\frac{A_z}{A_X}) 16_2\ e^{2i\phi_2} \\ W_{13}^N &= \lambda_H v_{10} \frac{v_{10}}{\bar{M}_P} \epsilon_P^6 \epsilon_G^2 16_1 (\frac{A_z}{v_5})(\frac{\bar{\Phi}}{v_{10}})(\frac{\bar{\Phi}}{v_{10}})(\frac{A_u}{v_5}) 16_3\ e^{i(\delta_\nu + \phi_3)} \\ W_{33}^N &= \lambda_H v_{10} \frac{v_{10}}{\bar{M}_P} \epsilon_P^8 \epsilon_G^2 16_3 (\frac{A_u}{v_5})^2 (\frac{A_z}{v_5})(\frac{\bar{\Phi}}{v_{10}})(\frac{\bar{\Phi}}{v_{10}})(\frac{A_u}{v_5})^2 16_3\ e^{2i\phi_3} \end{aligned}$$

It is then not difficult to read off the Clebsch factors

$$y_N = 9/25, \qquad z_N = 4, \qquad w_N = 256/27 \tag{70}$$

where $\delta_\nu$, $\phi_2$ and $\phi_3$ are three phases. For convenience , we first redefine the phases of the three right-handed neutrinos $\nu_{R1} \to e^{i\delta_\nu}\nu_{R1}$, $\nu_{R2} \to e^{i\phi_2}\nu_{R2}$, and $\nu_{R3} \to e^{i\phi_3}\nu_{R3}$, so that the matrix $M_N^G$ becomes real.

The light neutrino mass matrix is then given via see-saw mechanism as follows

$$\begin{aligned} M_\nu &= \Gamma_\nu^G (M_N^G)^{-1} (\Gamma_\nu^G)^\dagger v_2^2 / 2 R_t^{-6} \eta_N^2 \\ &= M_0 \begin{pmatrix} -\frac{1}{4}\frac{z_\nu}{w_\nu} z_N \epsilon_P^4 & -\frac{15}{2}\frac{y_\nu}{w_\nu} z_N \epsilon_P^2 \epsilon_G^2 e^{i\frac{\pi}{2}} & -\frac{\sqrt{1}}{4}\frac{x_\nu}{w_\nu} z_N \epsilon_P^2 \epsilon_G^2 \\ -\frac{15}{2}\frac{y_\nu}{w_\nu} z_N \epsilon_P^2 \epsilon_G^2 e^{-i\frac{\pi}{2}} & 15\frac{z_\nu}{w_\nu}\frac{w_N}{z_N}\epsilon_P^4 - \frac{x_\nu}{w_\nu}\epsilon_G^2 \cos\delta_\nu + \frac{15 y_\nu^2}{z_\nu w_\nu} z_N \epsilon_G^4 & e^{i\delta_\nu} + \frac{y_\nu x_\nu}{z_\nu w_\nu} z_N \epsilon_G^4 i \\ -\frac{\sqrt{1}}{4}\frac{x_\nu}{w_\nu} z_N \epsilon_P^2 \epsilon_G^2 & e^{-i\delta_\nu} - \frac{y_\nu x_\nu}{z_\nu w_\nu} z_N \epsilon_G^4 i & \frac{1}{60}\frac{x_\nu^2}{z_\nu w_\nu} z_N \epsilon_G^4 \end{pmatrix} \end{aligned}$$

$$= 2.45 \begin{pmatrix} 1.027\lambda^5 & -0.97\lambda^7 e^{i\frac{\pi}{2}} & 1.51\lambda^9 \\ -0.97\lambda^7 e^{-i\frac{\pi}{2}} & 0.37\lambda^4 \cos\delta_\nu - 0.535\lambda^4 - 0.92\lambda^9 & e^{i\delta_\nu} - 1.44\lambda^{10} e^{i\frac{\pi}{2}} \\ 1.51\lambda^9 & e^{-i\delta_\nu} - 1.44\lambda^{10} e^{-i\frac{\pi}{2}} & 0.49\lambda^{12} \end{pmatrix} \tag{71}$$

with

$$\begin{aligned} M_0 &= \left(\frac{2}{15^5}\right)^2 \left(\frac{15}{\epsilon_P^5}\right) \left(\frac{-w_\nu z_\nu}{y_N z_N}\right) \left(\frac{v_2^2}{2v_5}\right) R_t^{-6} \eta_N^2 \lambda_H \\ &= 2.45 \ \left(\frac{2.36 \times 10^{16} GeV}{v_5}\right) \left(\frac{\zeta_t}{1.04}\right) \ eV \end{aligned} \tag{72}$$

It is seen that only one phase, $\delta_\nu$ , is physical . We shall assume again maximum CP violation with $\delta_\nu = \pi/2$ . Neglecting the small terms of order above $O(\lambda^7)$, the neutrino mass matrix can be simply diagonalized by

$$V_\nu = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_\nu & -s_\nu \\ 0 & s_\nu & c_\nu \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\delta_\nu} \end{pmatrix} \tag{73}$$

and the charged lepton mass matrix by

$$V_e = \begin{pmatrix} \bar{c}_1 & -\bar{s}_1 & 0 \\ \bar{s}_1 & \bar{c}_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} i & 0 & 0 \\ 0 & c_e & -s_e \\ 0 & s_e & c_e \end{pmatrix} \tag{74}$$

The CKM-type lepton mixing matrix is then given by

$$\begin{aligned} V_{LEP} = V_\nu V_e^\dagger &= \begin{pmatrix} V_{\nu_e e} & V_{\nu_e \mu} & V_{\nu_e \tau} \\ V_{\nu_\mu e} & V_{\nu_\mu \mu} & V_{\nu_\mu \tau} \\ V_{\nu_\tau e} & V_{\nu_\tau \mu} & V_{\nu_\tau \tau} \end{pmatrix} \\ &= \begin{pmatrix} \bar{c}_1 & \bar{s}_1 & 0 \\ -\bar{s}_1(c_\nu c_e + s_\nu s_e e^{i\delta_\nu}) & \bar{c}_1(c_\nu c_e + s_\nu s_e e^{i\delta_\nu}) & -(s_\nu c_e - c_\nu s_e e^{i\delta_\nu}) \\ -\bar{s}_1(s_\nu c_e - c_\nu s_e e^{i\delta_\nu}) & \bar{c}_1(s_\nu c_e - c_\nu s_e e^{i\delta_\nu}) & c_\nu c_e + s_\nu s_e e^{i\delta_\nu} \end{pmatrix} \end{aligned} \tag{75}$$

where the angles are found to be

$$\tan\bar{\theta}_1 = \sqrt{\frac{m_e}{m_\mu}} = 0.0695 \tag{76}$$

$$\tan\theta_e = -\frac{x_e}{2w_e}\epsilon_G^2 = -\frac{m_\mu}{m_\tau}\frac{x_e}{6y_e} = 0.0149 \tag{77}$$

$$\tan\theta_\nu = 1 \tag{78}$$

For masses of light Majorana neutrinos we have

$$m_{\nu_e} = -\frac{1}{4}\frac{z_\nu}{w_\nu} z_N M_0 = 1.27 \times 10^{-3}\ eV, \tag{79}$$

$$m_{\nu_\mu} = \left(1 + \frac{15}{2}\frac{z_\nu w_N}{w_\nu z_N}\epsilon_P^4\right) M_0 \simeq 2.448464\ eV \tag{80}$$

$$m_{\nu_\tau} = \left(1 - \frac{15}{2}\frac{z_\nu w_N}{w_\nu z_N}\epsilon_P^4\right) M_0 \simeq 2.451536\ eV \tag{81}$$

The three heavy Majorana neutrinos have masses

$$M_{N_1} \simeq M_{N_3} \simeq \frac{1}{2} y_N z_N \epsilon_P^7 v_5 \lambda_H \simeq 333\ \left(\frac{v_5}{2.36\times 10^{16} GeV}\right)\ GeV \tag{82}$$

$$M_{N_2} = y_N \epsilon_P^5 v_5 \lambda_H = 1.63 \times 10^6\ \left(\frac{v_5}{2.36\times 10^{16} GeV}\right)\ GeV \tag{83}$$

The three heavy Majorana neutrinos in the present model have their masses much below the GUT scale, unlike many other GUT models with corresponding masses near the GUT scale. In fact, two of them have masses in the range comparable with the electroweak scale.

As the masses of the three light neutrinos are very small, a direct measurement for their masses would be too difficult. An efficient detection on light neutrino masses can be achieved through their oscillations. The probability that an initial $\nu_\alpha$ of energy $E$ (in unit MeV) gets converted to a $\nu_\beta$ after travelling a distance $L$ (in unit $m$) is

$$P_{\nu_\alpha\nu_\beta} = \delta_{\alpha\beta} - 4\sum_{j>i} V_{\alpha i}V_{\beta i}^* V_{\beta j} V_{\alpha j}^* \sin^2\left(\frac{1.27 L \Delta m_{ij}^2}{E}\right) \tag{84}$$

with $\Delta m_{ij}^2 = m_j^2 - m_i^2$ (in unit $eV^2$). From the above results, we observe the following

1. a $\nu_\mu(\bar{\nu}_\mu) \to \nu_e(\bar{\nu}_e)$ short wave-length oscillation with

$$\Delta m_{e\mu}^2 = m_{\nu_\mu}^2 - m_{\nu_e}^2 \simeq 6\ eV^2, \qquad \sin^2 2\theta_{e\mu} \simeq 1.0\times 10^{-2}\ , \tag{85}$$

which is consistent with the LSND experiment [57]

$$\Delta m_{e\mu}^2 = m_{\nu_\mu}^2 - m_{\nu_e}^2 \simeq (4-6)eV^2 \ , \qquad \sin^2 2\theta_{e\mu} \simeq 1.8 \times 10^{-2} \sim 3 \times 10^{-3} \ ; \tag{86}$$

2. a $\nu_\mu(\bar{\nu}_\mu) \to \nu_\tau(\bar{\nu}_\tau)$ long-wave length oscillation with

$$\Delta m_{\mu\tau}^2 = m_{\nu_\tau}^2 - m_{\nu_\mu}^2 \simeq 1.5 \times 10^{-2} eV^2 \ , \qquad \sin^2 2\theta_{\mu\tau} \simeq 0.987 \ , \tag{87}$$

which could explain the atmospheric neutrino deficit [58]:

$$\Delta m_{\mu\tau}^2 = m_{\nu_\tau}^2 - m_{\nu_\mu}^2 \simeq (0.5-2.4) \times 10^{-2} eV^2 \ , \qquad \sin^2 2\theta_{\mu\tau} \simeq 0.6 - 1.0 \ , \tag{88}$$

with the best fit [58]

$$\Delta m_{\mu\tau}^2 = m_{\nu_\tau}^2 - m_{\nu_\mu}^2 \simeq 1.6 \times 10^{-2} eV^2 \ , \qquad \sin^2 2\theta_{\mu\tau} \simeq 1.0 \ ; \tag{89}$$

3. Two massive neutrinos $\nu_\mu$ and $\nu_\tau$ with

$$m_{\nu_\mu} \simeq m_{\nu_\tau} \simeq 2.45 \ eV \ , \tag{90}$$

fall in the range required by possible hot dark matter [59].

4. $(\nu_\mu - \nu_\tau)$ oscillation will be beyond the reach of CHORUS/NOMAD and E803. However, $(\nu_e - \nu_\tau)$ oscillation may become interesting as a short wave-length oscillation with

$$\Delta m_{e\tau}^2 = m_{\nu_\tau}^2 - m_{\nu_e}^2 \simeq 6 \ eV^2, \qquad \sin^2 2\theta_{e\tau} \simeq 1.0 \times 10^{-2} \ , \tag{91}$$

which should provide an independent test on the pattern of the present Majorana neutrino mass matrix.

5. Majorana neutrino allows neutrinoless double beta decay $(\beta\beta_{0\nu})$ [60]. Its decay amplitude is known to depend on the masses of Majorana neutrinos $m_{\nu_i}$ and the lepton mixing matrix elements $V_{ei}$. The present model is compatible with the present experimental upper bound on neutrinoless double beta decay

$$\bar{m}_{\nu_e} = \sum_{i=1}^{3} [V_{ei}^2 m_{\nu_i} \zeta_i] \simeq 1.18 \times 10^{-2} \ eV \ < \ \bar{m}_\nu^{upper} \simeq 0.7 \ eV \tag{92}$$

The decay rate is found to be

$$\Gamma_{\beta\beta} \simeq \frac{Q^5 G_F^4 \bar{m}_{\nu_e}^2 p_F^2}{60\pi^3} \simeq 1.0 \times 10^{-61} GeV \tag{93}$$

with the two electron energy $Q \simeq 2$ MeV and $p_F \simeq 50$ MeV.

6. In this case, solar neutrino deficit has to be explained by oscillation between $\nu_e$ and a sterile neutrino $\nu_s$ [61,16,19]. Since strong bounds on the number of neutrino species both from the invisible $Z^0$-width and from primordial nucleosynthesis [62,63] require the additional neutrino to be sterile (singlet of SU(2)× U(1), or singlet of SO(10) in the GUT SO(10) model). Masses and mixings of the triplet sterile neutrinos can be chosen by introducing an additional singlet scalar with VEV $v_s \simeq 336$ GeV. We find

$$\begin{aligned} m_{\nu_s} &= \lambda_H v_s^2 / v_{10} \simeq 2.8 \times 10^{-3} eV \\ \sin\theta_{es} &\simeq \frac{m_{\nu_L \nu_s}}{m_{\nu_s}} = \frac{v_2}{2v_s} \frac{\epsilon_P}{\epsilon_G^2} \simeq 3.8 \times 10^{-2} \end{aligned} \tag{94}$$

with the mixing angle consistent with the requirement necessary for primordial nucleosynthesis [64] given in [62]. The resulting parameters

$$\Delta m_{es}^2 = m_{\nu_s}^2 - m_{\nu_e}^2 \simeq 6.2 \times 10^{-6} eV^2, \qquad \sin^2 2\theta_{es} \simeq 5.8 \times 10^{-3} \tag{95}$$

are consistent with the values [61] obtained from fitting the experimental data:

$$\Delta m_{es}^2 = m_{\nu_s}^2 - m_{\nu_e}^2 \simeq (4-9) \times 10^{-6} eV^2, \qquad \sin^2 2\theta_{es} \simeq (1.6-14) \times 10^{-3} \tag{96}$$

This scenario can be tested by the next generation solar neutrino experiments in Sudhuray Neutrino Observatory (SNO) and Super-kamiokanda (Super-K), both planning to start operation in 1996. From measuring neutral current events, one could identify $\nu_e \to \nu_s$ or $\nu_e \to \nu_\mu(\nu_\tau)$ since the sterile neutrinos have no weak gauge interactions. From measuring seasonal variation, one can further distinguish the small-angle MSW [65] oscillation from vacuum mixing oscillation.

## V. DIHEDRAL GROUP $\Delta(48)$

For completeness, we present in this section some features of the non-Abelian discrete dihedral group $\Delta(3n^2)$, a subgroup of SU(3). The generators of the $\Delta(3n^2)$ group consist of

the matrices

$$E(0,0) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \tag{97}$$

and

$$A_n(p,q) = \begin{pmatrix} e^{i\frac{2\pi}{n}p} & 0 & 0 \\ 0 & e^{i\frac{2\pi}{n}q} & 0 \\ 0 & 0 & e^{-i\frac{2\pi}{n}(p+q)} \end{pmatrix} \tag{98}$$

It is clear that there are $n^2$ different elements $A_n(p,q)$ since if $p$ is fixed, $q$ can take on $n$ different values. There are three different types of elements

$$A_n(p,q), \qquad E_n(p,q) = A_n(p,q)E(0,0), \qquad C_n(p,q) = A_n(p,q)E^2(0,0)$$

in the $\Delta(3n^2)$ group, therefore the order of the $\Delta(3n^2)$ group is $3n^2$. The irreducible representations of the $\Delta(3n^2)$ groups consist of i) $(n^2-1)/3$ triplets and three singlets when $n/3$ is not an interger and ii) $(n^2-3)/3$ triplets and nine singlets when $n/3$ is an interger.

The characters of the triplet representations can be expressed as [66]

$$\begin{aligned} &\Delta_T^{m_1 m_2}(A_n(p,q)) = e^{i\frac{2\pi}{n}[m_1 p + m_2 q]} + e^{i\frac{2\pi}{n}[m_1 q - m_2(p+q)]} + e^{i\frac{2\pi}{n}[-m_1(p+q)+m_2 p]} \\ &\Delta_T^{m_1 m_2}(E_n(p,q)) = \Delta_T^{m_1 m_2}(C_n(p,q)) = 0 \end{aligned} \tag{99}$$

with $m_1, m_2 = 0, 1, \cdots, n-1$. Note that $(-m_1+m_2, -m_1)$ and $(-m_2, m_1-m_2)$ are equivalent to $(m_1, m_2)$.

One will see that $\Delta(48)$ is the smallest of the dihedral group $\Delta(3n^2)$ with sufficient triplets for constructing interesting texture structures of the Yukawa coupling matrices.

The irreducible triplet representations of $\Delta(48)$ consist of two complex triplets $T_1 = (x, y, z)$, $\bar{T}_1 = (\bar{x}, \bar{y}, \bar{z})$ and $T_3 = (\alpha, \beta, \gamma)$, $\bar{T}_3 = (\bar{\alpha}, \bar{\beta}, \bar{\gamma})$, one real triplet $T_2 = \bar{T}_2 = (a, b, c)$ as well as three singlet representations. For a similar consideration as in Ref. [13] for $\Delta(75)$, the basis of the triplet representations of $\Delta(48)$ is chosen as

$$T_1 \otimes T_1 \mid_{T_2} = \begin{bmatrix} x^2 \\ y^2 \\ z^2 \end{bmatrix} \tag{100}$$

$$T_1 \otimes \bar{T}_1 \mid_{T_3} = \begin{bmatrix} y\bar{z} \\ z\bar{x} \\ x\bar{y} \end{bmatrix} \tag{101}$$

Thus the generator $\hat{E}(0,0)$ has the same representation matrix $D_R(\hat{E}(0,0))$ for all of the triplet representations $R$:

$$D_R(\hat{E}(0,0)) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad R\{T_1, \bar{T}_1, T_2, T_3, \bar{T}_3\} \tag{102}$$

The representation matrices corresponding to the generator $\hat{A}_4(1,0)$ are given by

$$\begin{aligned} D_1(\hat{A}_4(1,0)) &= A_4(1,0) = \begin{pmatrix} i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -i \end{pmatrix}, \\ D_{\bar{1}}(\hat{A}_4(1,0)) &= \bar{A}_4(1,0) = \begin{pmatrix} -i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i \end{pmatrix}, \\ D_2(\hat{A}_4(1,0)) &= A_4(2,0) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \\ D_3(\hat{A}_4(1,0)) &= A_4(1,2) = \begin{pmatrix} i & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & i \end{pmatrix}, \end{aligned} \tag{103}$$

$$D_{\bar{3}}(\hat{A}_4(1,0)) = \bar{A}_4(1,2) = \begin{pmatrix} -i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -i \end{pmatrix}$$

where $D_i$ is the representation matrix for the triplet $T_i$ and $D_{\bar{i}}$ for $\bar{T}_i$. The $A_4(p,q)$ matrices are defined in eq.(99). With the above basis and representations, one can explicitly construct the invariant tensors.

**Table 4**, Decomposition of the product of two triplets, $T_i \otimes T_j$ and $T_i \otimes \bar{T}_j$ in $\Delta(48)$. Triplets $T_i$ and $\bar{T}_i$ are simply denoted by $i$ and $\bar{i}$ respectively. For example $T_1 \otimes \bar{T}_1 = A \oplus T_3 \oplus \bar{T}_3 \equiv A3\bar{3}$, here $A$ represents singlets.

| $\Delta(48)$ | 1 | $\bar{1}$ | 2 | 3 | $\bar{3}$ |
|---|---|---|---|---|---|
| 1 | $\bar{1}\bar{1}2$ | A$3\bar{3}$ | $\bar{1}3\bar{3}$ | 123 | $12\bar{3}$ |
| 2 | $\bar{1}3\bar{3}$ | $13\bar{3}$ | A22 | $1\bar{1}\bar{3}$ | $1\bar{1}3$ |
| 3 | 123 | $\bar{1}23$ | $1\bar{1}\bar{3}$ | $2\bar{3}\bar{3}$ | A$1\bar{1}$ |

It is seen that each $T_i \otimes \bar{T}_i$ (i=1,2,3) contains all three singlet representations

$$\begin{aligned} T_i \otimes \bar{T}_i \mid_{A_1} &= x\bar{x} + y\bar{y} + z\bar{z}, \\ T_i \otimes \bar{T}_i \mid_{A_2} &= x\bar{x} + \omega y\bar{y} + \omega^2 z\bar{z}, \\ T_i \otimes \bar{T}_i \mid_{\bar{A}_2} &= x\bar{x} + \omega^2 y\bar{y} + \omega z\bar{z}, \end{aligned} \tag{104}$$

From table 4, one can obtain easily the structure of all three-triplet invariants. Following a similar consideration as in [13] for $\Delta(75)$, the three triplet invariant (ABC) can be specified by three numbers $\{ijk\}$ due to the property of the matrix representation under cyclic permutation in eq.(103), i.e., $(ABC) = A_iB_jC_k + c.p. = \{ijk\}$, where c.p. represent cyclic permutation of each representation's index. As an example, $\{112\} = (ABC) = (A_1B_1C_2 + A_2B_2C_3 + A_3B_3C_1)$. The product of three same triplets always contains two invariants

$$(T_iT_iT_i) = \{123\} + \{213\} \tag{105}$$

The remaining five independent invariants with three triplets are

$$\begin{aligned}
&\{111\}: \ (112); \qquad \{112\}: \ (1\bar{3}2); \\
&\{113\}: \ (132); \qquad \{123\}: \ (3\bar{1}1), \ (1\bar{3}3).
\end{aligned} \tag{106}$$

With the above structure, if one wants to find, for example, the invariant of the product $T_1 \otimes T_3 \otimes T_2$, one notes that $(T_1 T_2 T_3)$ is an invariant of the $\{113\}$ type, thus $T_1 \otimes T_3 \otimes T_2 \,|_{A1} = x\alpha c + y\beta a + z\gamma b$. Similarly, to find the $T_3$ contained in $\bar{T}_1 \otimes \bar{T}_2$, one yields from the same $\{113\}$ that

$$\bar{T}_2 \otimes \bar{T}_1 \,|_{T_3} = \begin{bmatrix} \bar{\gamma}\bar{x} \\ \bar{\alpha}\bar{y} \\ \bar{\beta}\bar{z} \end{bmatrix} \tag{107}$$

## VI. SUPERPOTENTIAL FOR FERMION YUKAWA INTERACTIONS

From the above properties of the dihedral group $\Delta(48)$, we can now construct the model in details. All three families with $3 \times 16 = 48$ chiral fermions are unified into a triplet 16-dimensional spinor representation of SO(10)$\times\Delta(48)$. Without losing generality, one can assign the three chiral families into one triplet representation $T_1$, which may be simply denoted as $\hat{16} = (16, T_1)$. All the fermions are assumed to obtain their masses through a single $10_1$ of SO(10) into which the needed two Higgs doublets are unified. It is possible to have a triplet sterile neutrino with small mixings with the ordinary neutrinos. A singlet scalar near the electroweak scale is necessary to generate small masses for the sterile neutrinos.

The superpotentials which lead to the above texture structures (eqs. (2)-(4) and (69)) with zeros and effective operators (eqs. (6) and (70)) are found to be

$$\begin{aligned}
W_Y = &\sum_{a=0}^{4} \psi_{a1} 10_1 \psi_{a2} + \bar{\psi}_{11}\chi_1\psi_{n+1} + \bar{\psi}_{12}\chi\psi_{n+1} + \bar{\psi}_{21}\chi_2\psi_{13} + \bar{\psi}_{22}\chi\psi_{13} \\
&+\bar{\psi}_{13}A_z\psi_{n+1} + \bar{\psi}_{31}\chi_3\psi_{23} + \bar{\psi}_{32}\chi\psi_{23} + \bar{\psi}_{23}A_u\psi_n + \bar{\psi}_{01}\chi_0\psi_{33} + \bar{\psi}_{02}\chi\psi_{33} \\
&+\bar{\psi}_{33}S'_G\psi_{n-3} + \bar{\psi}_{41}\chi_0\psi_{03} + \bar{\psi}_{42}\chi\psi_{43} + \bar{\psi}_{43}S_I\psi_{23} + \bar{\psi}_{03}S_I\psi_{13}
\end{aligned} \tag{108}$$

$$
+\bar{\psi}_0 S_G \hat{16} + \sum_{j=1}^{n+1} \bar{\psi}_j S_I \psi_{j-1} + \sum_{a=0}^{4}\sum_{i=1}^{2} S_G \bar{\psi}_{ai}\psi_{ai}
+ \sum_{i=1}^{2} \bar{\psi}_{i3} A_X \psi_{i3} + \sum_{j=1}^{n+1} \bar{\psi}_j A_X \psi_j + S_P(\bar{\psi}_{43}\psi_{43} + \bar{\psi}_{33}\psi_{33} + \bar{\psi}_{03}\psi_{03})
$$

for the fermion Yukawa coupling matrices,

$$
\begin{aligned}
W_R = & \sum_{i=1}^{3}(\psi_{i1}'^T \bar{\Phi} N_{i1} + N_{i2}^T \bar{\Phi}^T \psi_{i2}' + N_{i1}^T N_{i2} X_P + \bar{\psi}_{i1}' \chi_i' \psi_{i3}') + \sum_{i=1}^{2} \bar{\psi}_{i2}' \chi' \tilde{\psi}_{i3} \\
& + \bar{\psi}_{32}' \chi' \psi_{33}' + \bar{\psi}_{13}' A_z \psi_2' + \bar{\psi}_{23}' A_z \psi_1' + \bar{\psi}_{33}' A_z \psi_0' + \bar{\tilde{\psi}}_{13} S_G \psi_3' \\
& + \bar{\tilde{\psi}}_{13} A_u \psi_2' + \bar{\psi}_3' A_u \psi_2' + \bar{\psi}_2' A_u \psi_0' + \bar{\psi}_0' S_G' \hat{16} + \sum_{i=1}^{2} \bar{\psi}_{3i}' A_X \psi_{3i}' \\
& + \sum_{i=1}^{2}\sum_{j=1}^{2} S_I \bar{\psi}_{ij}' \psi_{ij}' + \sum_{i=1}^{3}(S_P' N_{i1}^T N_{i1} + S_P'' N_{i2}^T N_{i2}) \\
& + S_P(\sum_{i=1}^{3} \bar{\psi}_{i3}' \psi_{i3}' + \sum_{i=1}^{2} \bar{\tilde{\psi}}_{i3} \tilde{\psi}_{i3} + \sum_{a=0}^{3} \bar{\psi}_a' \psi_a')
\end{aligned} \tag{109}
$$

for the right-handed Majorana neutrinos, and

$$
W_S = \psi_1' 10_1 \psi_2' + \bar{\psi}_1' \Phi \nu_s + \bar{\psi}_2' \phi_s \hat{16} + (\bar{\nu}_s \phi_s N_s + h.c.) + S_I \bar{N}_s N_s \tag{110}
$$

for the sterile neutrino masses and their mixings with the ordinary neutrinos.

In the above superpotentials, each term is ensured by the U(1) symmetry. An appropriate assignment of U(1) charges for the various fields is implied. All $\psi$ fields are triplet 16-dimensional spinor heavy fermions. Where the fields $\psi_{a3}\{\bar{\psi}_{a3}\}$, $(a = 0, 1, 2, 3, 4)$, $\psi_{i3}'\{\bar{\psi}_{i3}'\}$, $(i = 1, 2, 3)$, $\tilde{\psi}_{i3}\{\bar{\tilde{\psi}}_{i3}\}$, $(i = 1, 2)$, $\psi_i\{\bar{\psi}_i\}$ $(i = 0, 1, \cdots n + 1)$, $\psi_i'\{\bar{\psi}_i'\}$, $(i = 0, 1, 2, 3)$ belong to $(16, T_1)\{(\bar{16}, \bar{T}_1)\}$ representations of SO(10) $\times\Delta(48)$; $\psi_{11}\{\bar{\psi}_{11}\}$ and $\psi_{12}\{\bar{\psi}_{12}\}$ belong to $(16, T_2)\{(\bar{16}, T_2)\}$; $\psi_{i1}\{\bar{\psi}_{i1}\}$ and $\bar{\psi}_{i2}\{\psi_{i2}\}$ $(i = 0, 2, 3, 4)$ belong to $(16, T_3)\{(\bar{16}, \bar{T}_3)\}$; $\psi_{i1}'\{\bar{\psi}_{i1}'\}$ and $\bar{\psi}_{i2}'\{\psi_{i2}'\}$ $(i = 1, 2)$ belong to $(16, \bar{T}_3)\{(\bar{16}, T_3)\}$; $\psi_{31}'\{\bar{\psi}_{31}'\}$ and $\psi_{32}'\{\bar{\psi}_{32}'\}$ belong to $(16, T_2)\{(\bar{16}, T_2)\}$; $N_{i2}\{N_{i1}\}$ $(i = 1, 2)$ belong to $(1,\ \bar{T}_3)\{1,\ T_3)\}$; $N_{31}\{N_{32}\}$ belong to $(1,\ T_2)\{(1,\ T_2)\}$; $S_G$, $S_G'$, $S_I$, $S_P$ and $\phi_s$ are singlet scalars of SO(10) $\times\Delta(48)$. $\nu_s$ and $N_s$ are SO(10) singlet and $\Delta(48)$ triplet fermions. The SO(10) singlets $N_{i1}$ and $N_{i2}$ (i=1,2,3) are $\Delta(48)$ triplets heavy Majorana fermions above the GUT scale. They are introduced to generate the right-handed Majorana neutrino masses and mixings in the SO(10)

grand unified models if the **126**-dimensional representation Higgs fields do not allow to exist in in a fundamental theory. Recently, it was shown in ref. [67] that for fermionic compactification schemes the **126**-dimensional representations appear unlikely to emerge from the compactification of heterotic string models. All SO(10) singlet $\chi$ fields are triplets of $\Delta(48)$. Where $(\chi_1, \chi_2, \chi_3, \chi_0, \chi)$ belong to triplet representations $(\bar{T}_3, T_3, \bar{T}_1, T_2, \bar{T}_3)$ respectively; $(\chi'_1, \chi'_2, \chi'_3, \chi')$ belong to triplet representations $(\bar{T}_1, T_2, T_3, T_3)$ respectively. With the above assignment for various fields, one can check that once the triplet field $\chi$ develops VEV only along the third direction, i.e., $<\chi^{(3)}>\neq 0$, and $\chi'$ develops VEV only along the second direction, i.e., $<\chi'^{(2)}>\neq 0$, the resulting fermion Yukawa coupling matrices at the GUT scale will automatically have, due to the special features of $\Delta(48)$, the interesting texture structure with four non-zero textures '33', '32', '22' and '12' characterized by $\chi_1$, $\chi_2$, $\chi_3$, and $\chi_0$ respectively, and the resulting right-handed Majorana neutrino mass matrix has three non-zero textures '33', '13' and '22' characterized by $\chi'_1$, $\chi'_2$, and $\chi'_3$ respectively. It is seen that five triplets are needed. Where one triplet is necessary for unification of the three family fermions, and four triplets are required for obtaining the needed minimal non-zero textures. In figures 1 and 2, we have illustrated the non-zero textures needed for the Dirac fermion Yukawa coupling matrices and Majorana mass matrix, respectively. To obtain the realistic fermion Yukawa coupling matrices and Majorana mass matrix, one uses the Froggatt-Nielsen mechanism [68] to understand the small mass ratios and an effective operator analysis to yield the appropriate Clebsch-Gordan coefficients.

The symmetry breaking scenario and the structure of the physical vacuum are considered as follows

$$\begin{aligned} & SO(10)\times\Delta(48)\times U(1) \stackrel{\bar{M}_P}{\rightarrow} SO(10) \stackrel{v_{10}}{\rightarrow} SU(5) \\ & \stackrel{v_5}{\rightarrow} SU(3)_c\times SU(2)_L\times U(1)_Y \stackrel{v_1,v_2}{\rightarrow} SU(3)_c\times U(1)_{em} \end{aligned} \tag{111}$$

and: $<S'_P = S''_P =<S_P>= X_P = \bar{M}_P$, $<S_I>=v_{10}$, $<\Phi^{(16)}>=<\bar{\Phi}^{(16)}>=v_{10}/\sqrt{2}$, $S_G =<S'_G>=v_5$ , $<\chi^{(3)}>=<\chi_a^{(i)}>=\bar{M}_P$, $<\chi'^{(2)}>=<\chi_j'^{(i)}>=v_5$ with $(i=1,2,3; a=0,1,2,3; j=1,2,3)$, $<\chi^{(1)}>=<\chi^{(2)}>=<\chi'^{(1)}>=<\chi'^{(3)}>=0$, $<\phi_s>=v_s\simeq 336$ GeV,

$< H_2 >= v_2 = v \sin\beta$ and $< H_1 >= v_1 = v \cos\beta$ with $v = \sqrt{v_1^2 + v_2^2} = 246$ GeV.

The superpotential for the Higgs sector will be presented elsewhere since it is also an important part as a complete model.

FIGURES

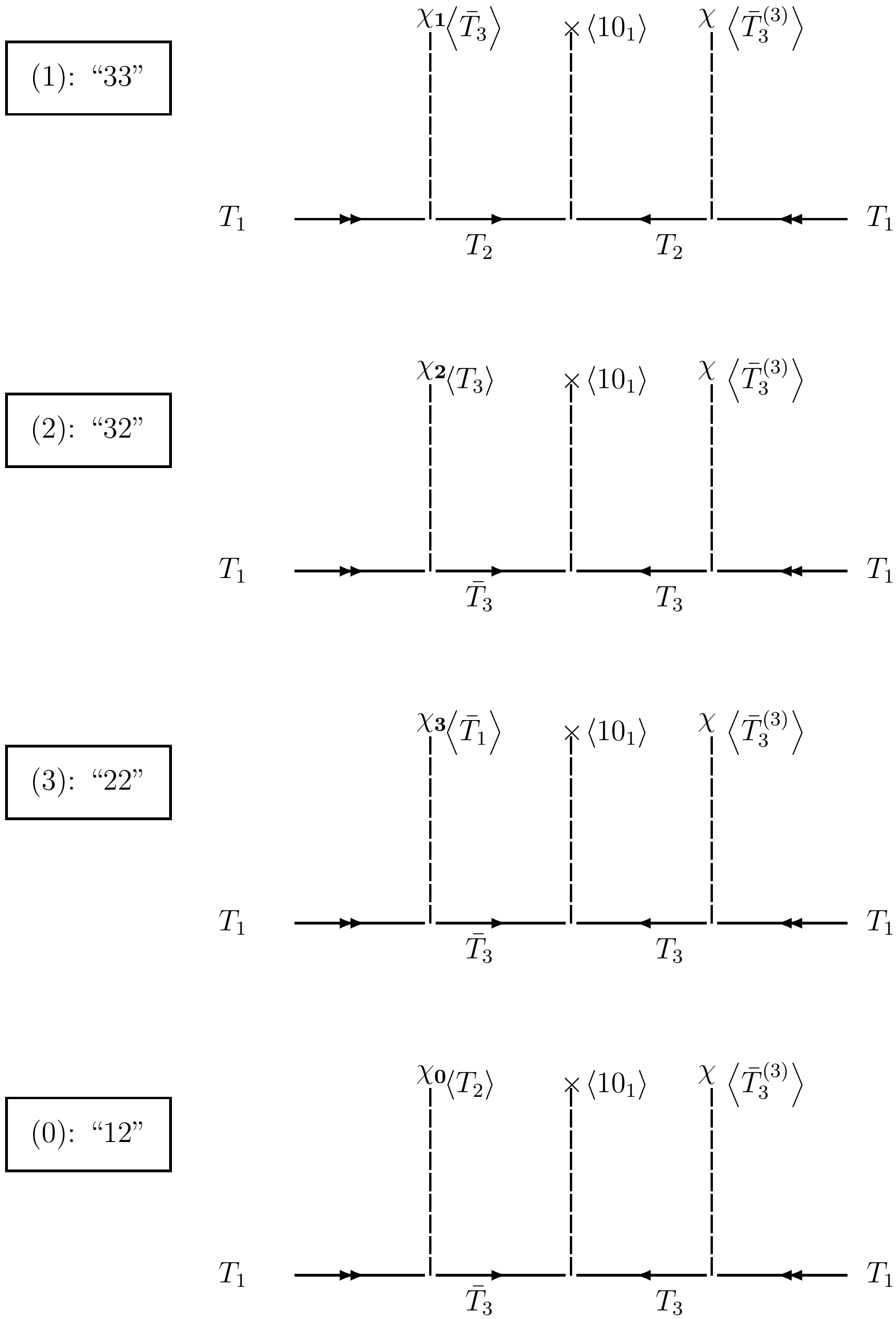


**Fig. 1.** Four non-zero Textures Resulting Naturally from the Family Symmetry $\Delta(48)$

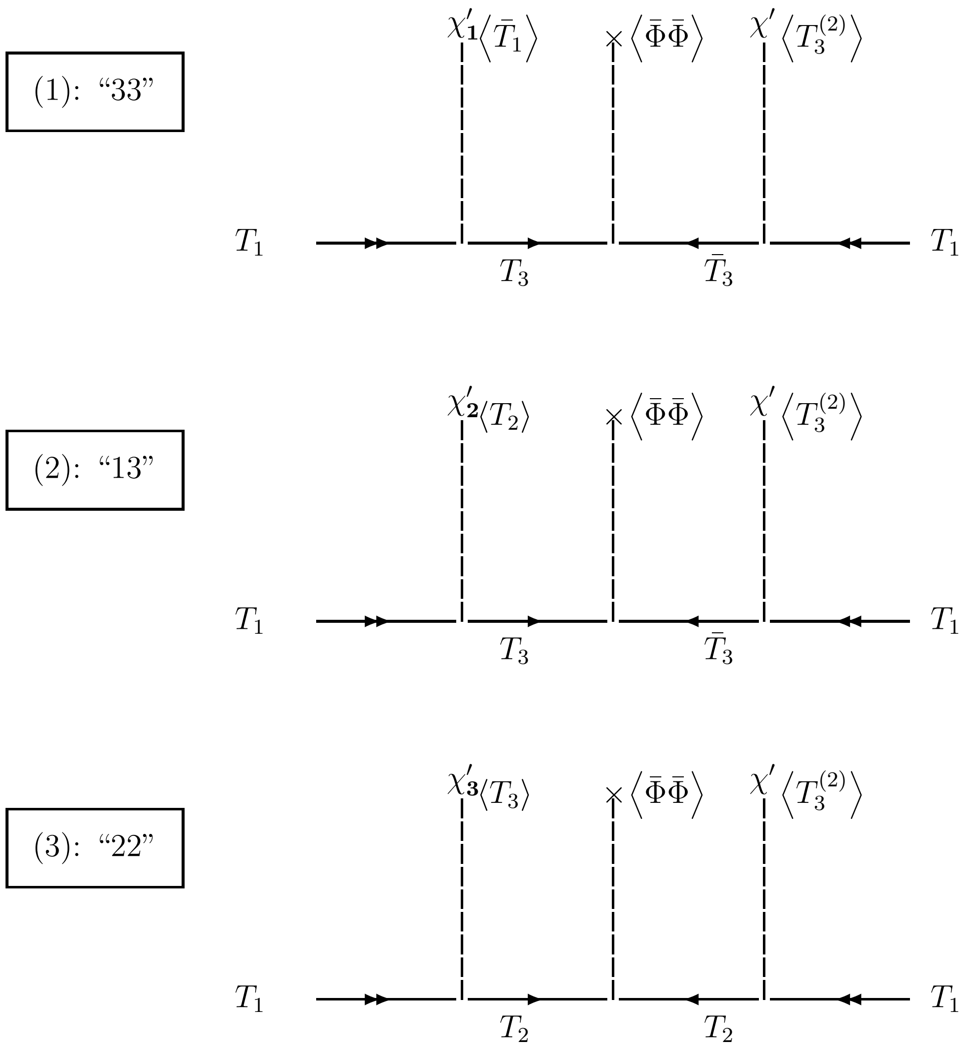


**Fig. 2.** Three non-zero textures for Majorana Neutrino Mass Matrix.

## VII. CONCLUSIONS

Based on the symmetry group SUSY SO(10)$\times\Delta(48)\times$ U(1), we have presented in much greater detail an alternative interesting model with small $\tan\beta$. It is amazing that nature has allowed us to make predictions in terms of a single Yukawa coupling constant and three ratios of the VEVs determined by the structure of the physical vacuum and understand the low energy physics from the GUT scale physics. It has also suggested that nature favors maximal spontaneous CP violation. In comparison with the model with large $\tan\beta \sim m_t/m_b$, i.e., Model I, the model analyzed here with low $\tan\beta$, i.e., Model II has provided a consistent picture on the 23 parameters with better accuracy. Besides, ten relations involving fermion masses and CKM matrix elements are obtained with four of them independent of the RG scaling effects. These relations are our main results which contain only low energy observables. As an analogy to the Balmer series formula, these relations may be considered as empirical at the present moment. They have been tested by the existing experimental data to a good approximation and can be tested further directly by more precise experiments in the future. The two types of the models corresponding to the large $\tan\beta$ (Model I) and low $\tan\beta$ (Model II) might be distinguished in testing the MSSM Higgs sector at Colliders as well as by precisely measuring the ratio $|V_{ub}/V_{cb}|$ since this ratio does not receive radiative corrections in both models. The neutrino sector is of special interest for further study. Though the recent LSND experiment, atmospheric neutrino deficit, and hot dark matter could be simultaneously explained in the present model, solar neutrino puzzle can be understood only by introducing an SO(10) singlet sterile neutrino. The scenario for the neutrino sector can be further tested through $(\nu_e - \nu_\tau)$ and $(\nu_\mu - \nu_\tau)$ oscillations since the present scenario has predicted a short wave $(\nu_e - \nu_\tau)$ oscillation. However, the $(\nu_\mu - \nu_\tau)$ oscillation is beyond the reach of CHORUS/NOMAD and E803. It is expected that more precise measurements from CP violation, neutrino oscillation and various low energy experiments in the near future could provide crucial tests on the present model and guide us to establish a more fundamental theory.

**Acknoledgement** Y.L. Wu would like to thank Institute of Theoretical Physics, Chinese Academy of Sciences, for its hospitality and partial support during his visiting.

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